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Mean-field renormalization group study of the long-range Ising model

K H Khoo and H K Sy

Department of Physics, National University of Singapore, 10 Kent Ridge Crescent, 119260 Singapore

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Abstract

In this paper, we study the one-dimensional long-range Ising model with a spin–spin interaction that decays as $1/r^{1+\sigma}$ using the mean-field renormalization group. The critical coupling K_c and critical exponents y_t and y_h are computed using cluster sizes of up to sixteen spins. We then apply the alternating-alpha VBS (Vanden Broeck–Schwartz) transformation to accelerate the convergence of results for different cluster sizes.

1. Introduction

The long-range Ising model with ferromagnetic interactions that decay as $1/r^{d+\sigma}$ has been studied for the past three decades and some of the most important results are presented here. For the 1D case, Dyson [1] has shown that a phase transition occurs when $0 < \sigma < 1$ and no phase transition occurs when $\sigma > 1$. For $\sigma = 1$, the magnetization is discontinuous at a finite temperature and this is known as the Thouless effect [2, 3]. Several estimates for the critical properties have also been obtained using finite-range scaling [4], Padé approximants [5], mean-field theory [6], perturbation theory [7], Bethe lattice approximation [8] and the coherent anomaly method [9]. In higher dimensions, Fisher, Ma and Nickel [10] have obtained critical exponents for general spin models with long-range interactions that decay as $1/r^{d+\sigma}$ using renormalization group expansions. The critical properties of the long-range Ising model in one, two and three dimensions have also been studied using Monte Carlo simulations by Luijten and Blöte [11–13] up to a relatively high accuracy.

There are several reasons for studying long-range models with interactions that decay as $1/r^{d+\sigma}$ and one is that interactions such as the van der Waals or dipole–dipole forces which occur commonly in nature decay as $1/r^{d+\sigma}$. It has also been claimed that screened Coulomb interactions in ionic systems lead to effectively $1/r^{d+\sigma}$ decaying interactions [14, 15] while critical exponents of the long-range universality class have been observed experimentally in a ferromagnetic phase transition [16]. Casimir forces between uncharged particles in a critical fluid arising from critical fluctuations have also been shown to exhibit $1/r^{d+\sigma}$ dependence [17] while Yuval and Anderson [18] have shown that the Kondo problem corresponds to the one-dimensional Ising model with a combination of inverse-square and nearest-neighbour interactions. Finally, these long-range models allow us to study phenomena above the upper critical dimension d_u in low-dimensional systems because d_u is σ-dependent and we can set $d > d_u$ by varying σ [11]. In this paper, we first calculate the critical properties of the 1D long-range Ising model using the mean-field renormalization group or MFRG for various cluster sizes. This is followed by an extrapolation using the VBS (Vanden Broeck–Schwartz) transformation to accelerate the convergence of the sequence. In the next few sections, we describe the MFRG and VBS transformation techniques used to investigate the critical properties of the long-range Ising model. We then present the results and our conclusions.

2. The MFRG method

Several years ago, a simple and versatile scheme known as the mean-field renormalization group or MFRG was introduced by Indekeu, Maritan and Stella [19,20]. This method combines the classical mean-field theory with the modern approach of the renormalization group mapping and it establishes an important link between the classical and modern theories of critical phenomena. Apart from its theoretical significance, the MFRG also has the advantage of easy and wide applicability and it has been used extensively to study a variety of problems such as those of geometrical critical phenomena and percolation [21,22], dynamic critical phenomena in Glauber models [23] and in quantum spin systems [24], the ANNNI model [25], mixed-spin systems [26] and the site–bond-correlated Ising model [27]. In this paper, we use the MFRG method to study the one-dimensional Ising model with long-range interactions that fall off as $1/r^{1+\sigma}$ described by the following reduced Hamiltonian:

$$H = -\beta\mathcal{H} = K\sum_{i,j}\frac{S_i S_j}{r_{ij}^{1+\sigma}} + h\sum_i S_i \tag{1}$$

where K indicates the strength of the ferromagnetic interaction, h is the reduced magnetic field and r_{ij} is the distance between the ith and jth sites.

In the mean-field renormalization group scheme, which is closely related to the phenomenological renormalization group [28], the approximate magnetization per spin is calculated for two clusters of different sizes N, N' using the mean-field approximation. The condition that the symmetry-breaking mean-field spins b, b' and the magnetizations per spin m_N, $m'_{N'}$ scale in the same way is then imposed and we have

$$m_{N'}(K', h', b') = l^{d-y_h} m_N(K, h, b) \tag{2}$$

$$b' = l^{d-y_h} b \tag{3}$$

where $l = (N/N')^{1/d}$ is the length scaling factor. If we carry out an expansion of m_N to first order in b, h and an expansion of $m'_{N'}$ to first order in b', h', we obtain the following:

$$m_N(K, h, b) = A_N(K)b + B_N(K)h \tag{4}$$

$$m_{N'}(K', h', b') = A_{N'}(K')b' + B_{N'}(K')h'. \tag{5}$$

Substituting equations (4), (5) into equation (2) and using equation (3) with $h' = L^{y_h}h$, we have

$$A_N(K) = A_{N'}(K') \tag{6}$$

$$B_N(K) = l^{2y_h-d} B_{N'}(K') \tag{7}$$

$$l^{y_t} = \left(\frac{\partial K'}{\partial K}\right)_{K=K_c} = \left(\frac{\partial A_N(K)}{\partial K}\right)_{K=K_c} \bigg/ \left(\frac{\partial A_{N'}(K')}{\partial K'}\right)_{K'=K_c}. \tag{8}$$

Equation (6) can be used to obtain the critical coupling K_c, which can then be substituted into equations (7) and (8) to yield the exponents y_t and y_h. Next, let us look at the procedure used to calculate the mean-field magnetization for the 1D long-range Ising model and assume the presence of an infinite number of symmetry-breaking mean-field spins on both sides of the cluster. The reduced cluster Hamiltonian is

$$H_N(K, h, b, S_i) = K \sum_{i,j=1,i>j}^{N} \frac{S_i S_j}{r_{ij}^{1+\sigma}} + h\sum_{i=1}^{N} S_i + Kb\sum_{i=1}^{N}\left(\sum_{k=N-i+1}^{\infty} \frac{1}{k^{1+\sigma}} + \sum_{l=i}^{\infty}\frac{1}{l^{1+\sigma}}\right) S_i \qquad (9)$$

where N denotes the number of cluster spins and the first term accounts for interactions between spins within the cluster. The second is the magnetic field term and the third term accounts for interactions between the mean-field spins and the cluster spins.

3. The VBS transformation

The next step is to provide the motivation for the application of the VBS transformations to the MFRG results and to do this we consider the finite-size scaling relation [29] for a system of linear dimension L in d dimensions:

$$f(t, h, K_3, \ldots, L^{-1}) = L^{-d} f(L^{y_t} t, L^{y_h} h, L^{y_3} K_3, \ldots, 1) \qquad (10)$$

where the K_n are the scaling fields and f is the singular part of the free energy per spin. If we differentiate equation (10) w.r.t. h, we get the finite-size scaling relation for the magnetization:

$$\begin{aligned} m(t, h, K_3, \ldots, L^{-1}) &= L^{y_h-d} m(L^{y_t} t, L^{y_h} h, L^{y_3} K_3, \ldots, 1) \\ &= L^{y_h-d} Y_m(L^{y_t} t, L^{y_h} h, L^{y_3} K_3, \ldots). \end{aligned} \qquad (11)$$

Next, let us define a renormalization between coupling constants for clusters of different sizes L and L' as follows:

$$L'^{y_n} K'_n = L^{y_n} K_n \qquad (12)$$

to apply for all coupling constants. Substitution of equation (12) into equation (11) yields

$$m(t, h, K_3, \ldots, L^{-1}) = \left(\frac{L}{L'}\right)^{y_h-d} m(t', h', K'_3, \ldots, L'^{-1}) \qquad (13)$$

which is exactly the same as equation (2); thus we can take equation (12) as the defining equation for the MFRG. It has also been shown that the critical properties obtained using the RG transformation of equation (12) are given by

$$T_c(L, L') = T_c + k_1 L^{y_3-1/\nu} + \cdots \qquad (14)$$

$$y_t(L, L') = y_t + k_2 L^{y_3} + \cdots \qquad (15)$$

$$y_h(L, L') = y_h + k_3 L^{y_3} + \cdots \qquad (16)$$

where y_3 is the exponent of the leading irrelevant variable K_3 and the cluster sizes are L and $L' = L - 1$ [30]. To show that the estimates of the critical properties are of the above form, we first expand the correlation length and its derivatives in powers of $L^{y_3} K_3$ at the critical point. Next, we express the critical properties in terms of the correlation length and its derivatives and substitute the expansions that we have obtained into these equations. Finally, we perform an expansion in powers of L on the expressions for the critical properties after the above substitutions to obtain equations (14) to (16).

From the above arguments, we see that the critical properties obtained from the MFRG have the form

$$X_L = X_\infty + C_{1L} L^{-x_1} + C_{2L} L^{-x_2} + \cdots \qquad (17)$$

where X_∞ is the true value of the critical property and X_L is the value obtained using clusters of sizes $L, L-1$. We now introduce an extrapolation technique known as the VBS transformations developed by Vanden Broeck and Schwartz [31] and we will see that it can be applied to improve our MFRG results. In this technique, we first have a sequence of approximants A_L which converge to a limiting value $\lim_{L\to\infty} A_L = A_\infty$. We then form a table of approximants $[L, M]$ by using the following equations:

$$[L, 0] = A_L \tag{18}$$

$$[L, -1] = \infty \tag{19}$$

$$\frac{1}{[L, M+1] - [L, M]} + \frac{\alpha_M}{[L, M-1] - [L, M]} = \frac{1}{[L+1, M] - [L, M]} + \frac{1}{[L-1, M] - [L, M]} \tag{20}$$

where the choice of α_M depends on the nature of the sequence. For approximants A_L which are of the form given in equation (17), Hamer and Barber [32] have shown that the appropriate form of α_M is given by

$$\alpha_M = -\frac{1 - (-1)^M}{2} \tag{21}$$

and, in our calculations, the initial approximants are generated by evaluating the critical properties using the MFRG. We then set the parameter L to be the length of the cluster and apply the transformations to generate the improved estimates. It can also be seen from equation (20) that the iterated values are dependent on the differences between the approximants; thus the sensitivity of the results to the round-off errors increases with every iteration and it is therefore necessary to perform the calculation up many more decimal places than are required in the final result. To estimate the error in the results, we utilize the 'L-shift' procedure [32] in which we multiply all of the initial approximants $[L, 0]$ by $1 + \varepsilon/L$ before performing the VBS transformation. This procedure is then repeated for various values of ε ranging from -1 to 1 and the stability of the final iterates $f(\varepsilon) = [L, M_f]$ with respect to ε is studied. It can be seen that such a multiplication retains the general form of equation (17) and this justifies the application of the VBS transformation to the new sequence. An optimal value of ε_o is then chosen which corresponds to a region where $f(\varepsilon)$ is reasonably stable with respect to changes in ε, and $f(\varepsilon_o)$ provides a good estimate of the value of the critical property while the variation of $f(\varepsilon)$ in this region provides a good measure of its error.

4. Results and discussion

Following the procedure in the previous section, we obtain approximants of K_c for $\sigma = 0.1$ by applying equations (30) and (31) to K_c-values obtained using the MFRG. We then list the approximants in table 1 up to 12 significant figures though the actual calculation was done with 20 significant figures. We then repeat the same procedure for the critical coupling K_c as well as the critical exponents y_t and y_h in the range $0.1 \leqslant \sigma \leqslant 0.9$ and the final results are presented in table 2 along with the results from other calculations.

In these two tables, 'FRS' represents values from finite-range scaling [4], 'PA' from Padé approximants [5], 'BL 1' and 'BL 2' from Bethe lattice approximation [8], 'PER' from perturbation theory [7], 'MFA' from mean-field theory [6] and 'CAM' from the coherent anomaly method [9]. Also, a number in brackets, for the Monte Carlo simulation results [11] and our results, represents the error in the last digit of the estimate.

It can be seen from table 2(a) that the VBS transformations yields K_c-values which are very close to the Monte Carlo simulation values. As for the critical exponent y_t, the extrapolated values show some deviation in the high and intermediate σ-regions, but it can be seen that the

Table 1. The approximant table for K_c at $\sigma = 0.1$.

L	$K_c = [L, 0]$	$[L, 1]$	$[L, 2]$	$[L, 3]$
2	0.0496219822605			
3	0.0491712323474	0.0485403468371		
4	0.0489083234386	0.0483783568836	0.0475794463221	
5	0.0487325922739	0.0482709667042	0.0475885877659	0.0476034029956
6	0.0486053134760	0.0481937081926	0.0475942409961	0.0476061126560
7	0.0485080966022	0.0481350269400	0.0475980705908	0.0476079190477
8	0.0484309762437	0.0480886963914	0.0476008279717	0.0476092188069
9	0.0483680369694	0.0480510391960	0.0476029033466	0.0476101992939
10	0.0483155240336	0.0480197318767	0.0476045191086	0.0476109646079
11	0.0482709283188	0.0479932281679	0.0476058110148	0.0476115781424
12	0.0482325032627	0.0479704557498	0.0476068664832	0.0476120781096
13	0.0481989920826	0.0479506456219	0.0476077441957	0.0476125051114
14	0.0481694651805	0.0479332307401	0.0476084852829	
15	0.0481432188080	0.0479177830436		
16	0.0481197095071			
L	$[L, 4]$	$[L, 5]$	$[L, 6]$	$[L, 7]$
6	0.0476160821863			
7	0.0476166721871	0.0476168914238		
8	0.0476168320287	0.0476169001460	0.0476168917180	
9	0.0476168797916	0.0476169937946	0.0476169587493	0.0476169237680
10	0.0476169134521	0.0476168837180	0.0476168855873	
11	0.0476166585428	0.0476168857495		
12	0.0476187492191			

results yielded by most methods also show problems when it comes to y_t and our results are already better than most of the other estimates. For the y_h-values, it can be seen that there is good agreement with the simulation results throughout the entire range except near $\sigma \sim 1$ and this could be due to the crossover to short-range behaviour. It can generally be seen that the results for the critical properties are very good in the region $\sigma \sim 0$ and this could be due to the fact that the MFRG method works better when the interaction is very long range. It can also be seen that the critical couplings are generally higher in accuracy than the exponents and this can again be attributed to the underlying MFRG approach which tends to yield better results for the critical coupling. We also see that the convergence rate of the approximants in the VBS transformation scheme tends to decrease with increasing σ in the region $0 < \sigma < 1/2$. We can explain this observation by substituting $y_t = \sigma$ and $y_3 = 2\sigma - d$ [11] into equations (14) to (16) to yield

$$T_c(L) \approx T_c + aL^{\sigma-d} \tag{22}$$

$$y_t(L) \approx y_t + bL^{2\sigma-d} \tag{23}$$

$$y_h(L) \approx y_h + cL^{2\sigma-d} \tag{24}$$

and we see that the convergence rate of each of the terms decreases with increasing σ as expected.

We can see from tables 2(a) to 2(c) that the VBS transformation produces estimates comparable in accuracy to the Monte Carlo simulation values for most values of σ, yet it requires significantly less computational resources. Thus we can obtain estimates of K_c, y_t and y_h to an accuracy higher than that of most existing estimates by simply carrying out the calculation with larger clusters. This is relatively straightforward as the only requirement for

Table 2. (a) K_c-estimates, (b) y_t-estimates and (c) y_h-estimates from various sources. (MC ≡ Monte Carlo.)

(a)

σ	VBS	PER	MFA	CAM	MC
0.1	0.04761693(20)	0.0481	0.04785	0.04777	0.0476168(6)
0.2	0.0922344(52)	—	0.09340	0.0928	0.0922314(15)
0.3	0.136099(91)	0.144	0.1385	0.1375	0.136110(2)
0.4	0.181113(20)	—	0.1841	0.183	0.181150(3)
0.5	0.229167(33)	0.250	0.2308	0.231	0.229155(6)
0.6	0.281831(62)	—	0.2791	0.282	0.281800(5)
0.7	0.341244(13)	0.391	0.3291	0.338	0.341237(4)
0.8	0.411060(77)	—	0.3811	0.401	0.411090(8)
0.9	0.499602(47)	0.65	0.4349	0.4728	0.49963(2)

σ	VBS	BL 1	BL 2	FRS	PA
0.1	0.04761693(20)	⩾0.04726	⩽0.09456	0.0505	—
0.2	0.0922344(52)	⩾0.08947	⩽0.1792	0.0923	0.0926
0.3	0.136099(91)	⩾0.1273	⩽0.2558	0.1362	0.1370
0.4	0.181113(20)	⩾0.1615	⩽0.3258	0.1815	0.1825
0.5	0.229167(33)	⩾0.1923	⩽0.3903	0.230	0.2307
0.6	0.281831(62)	⩾0.2203	⩽0.4502	0.282	0.2832
0.7	0.341244(13)	⩾0.2458	⩽0.5066	0.341	0.343
0.8	0.411060(77)	⩾0.2691	⩽0.5596	0.411	0.412
0.9	0.499602(47)	⩾0.2903	⩽0.6101	0.499	0.499

(b)

σ	VBS	FRS	PA	PER	CAM	Exact + MC
0.1	0.099951(43)	0.1096	—	0.09542	—	0.10
0.2	0.200022(65)	0.2041	—	—	—	0.20
0.3	0.29973(82)	0.2933	—	0.2564	—	0.30
0.4	0.3890(16)	0.3690	—	—	—	0.40
0.5	0.463(13)	0.4274	—	0.3559	—	0.50
0.6	0.5030(34)	0.4630	0.53	—	0.472	0.502(8)
0.7	0.50134(24)	0.4710	0.56	0.3759	0.461	0.491(10)
0.8	0.4553(30)	0.4529	0.53	—	0.446	0.457(10)
0.9	0.4093(32)	0.3802	0.50	0.2564	0.435	0.379(15)

(c)

σ	VBS	Exact + MC
0.1	0.550000(16)	0.55
0.2	0.600041(38)	0.60
0.3	0.650102(65)	0.65
0.4	0.70068(34)	0.70
0.5	0.75085(51)	0.75
0.6	0.8003(10)	0.798(4)
0.7	0.85085(51)	0.848(3)
0.8	0.90064(61)	0.896(4)
0.9	0.9446(15)	0.9508(10)

large-cluster calculations is more computational power and this is readily available. Another advantage of the VBS transformation is that it is one of the few methods that produces good estimates of the exponent y_h and our results for the 1D case support the conjecture that $y_h = (\sigma + d)/2$, first suggested in [10] and confirmed in [11, 13]. We can also extend the present calculation to higher-dimensional long-range spin systems and hope to generate high-accuracy results as in the 1D case as this is a relatively unexplored problem.

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